Name: ______________________________

Instructor: __________________________

Math 10560, Practice Exam 1.

- The Honor Code is in effect for this examination. All work is to be your own.
- No calculators.
- The exam lasts for 1 hour and 15 min.
- Be sure that your name is on every page in case pages become detached.
- Be sure that you have all 13 pages of the test.
- Each multiple choice question is worth 7 points. Your score will be the sum of the best 10 scores on the multiple choice questions plus your score on questions 13-16.

PLEASE MARK YOUR ANSWERS WITH AN X, not a circle!

1.	(a)	(b)	(c)	(d)	(e)
2.	(a)	(b)	(c)	(d)	(e)
3.	(a)	(b)	(c)	(d)	(e)
4.	(a)	(b)	(c)	(d)	(e)
5.	(a)	(b)	(c)	(d)	(e)
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11.	(a)	(b)	(c)	(d)	(e)
12.	(a)	(b)	(c)	(d)	(e)

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Multiple Choice	______________
13.	______________
14.	______________
15.	______________
Total	______________

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Multiple Choice

1.(7 pts.) The function $f(x) = x^3 + 3x + e^{2x}$ is one-to-one. Compute $(f^{-1})'(1)$.

- $(f^{-1})'(1) = \dfrac{1}{f'(f^{-1}(1))}$
- By trial-and-error we determine that $f^{-1}(1) = 0$ (that is $f(0) = 1$).
- $f'(x) = 3x^2 + 3 + 2e^{2x}$.
- Hence $f'(f^{-1}(1)) = f'(0) = 5$.
- Therefore $(f^{-1})'(1) = \dfrac{1}{5}$.

(a) $\dfrac{1}{6+e}$ (b) $\dfrac{1}{6+2e}$ (c) 0 (d) $\dfrac{1}{4}$ (e) $\dfrac{1}{5}$

2.(7 pts.) Solve the following equation for x:

$$\ln(x+4) - \ln x = 1 \ .$$

- Amalgamating the logarithms, our equation becomes:
$$\ln\left(\frac{x+4}{x}\right) = 1.$$
- Applying the exponential to both sides, we get
$$\left(\frac{x+4}{x}\right) = e^1 = e$$
- Multiplying both sides by x, we get $x + 4 = ex$ and $x - ex = -4$.
- Therefore $x(1-e) = -4$ and
$$x = \frac{-4}{1-e} = \frac{4}{e-1}.$$

(a) There is no solution. (b) $x = \dfrac{4}{e-1}$

(c) $x = \dfrac{4}{1-e}$ (d) $x = e+2$ and $x = e-2$

(e) $x = \dfrac{4}{e-1}$ and $x = \dfrac{4}{e+1}$

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3.(7 pts.) Differentiate the function

$$f(x) = \frac{(x^2-1)^4}{\sqrt{x^2+1}}.$$

- Use logarithmic differentiation. (Take logarithm of both sides of equation, then do implicit differentiation.)
- $y = \dfrac{(x^2-1)^4}{\sqrt{x^2+1}}$.
- $\ln y = 4\ln(x^2-1) - \dfrac{1}{2}\ln(x^2+1)$
- $\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{8x}{x^2-1} - \dfrac{x}{x^2+1}$.
- $f'(x) = \dfrac{dy}{dx} = \dfrac{x(x^2-1)^4}{\sqrt{x^2+1}}\left(\dfrac{8}{x^2-1} - \dfrac{1}{x^2+1}\right)$.

(a) $f'(x) = \dfrac{x(x^2-1)^4}{\sqrt{x^2+1}}\left(\dfrac{8}{x^2-1} + \dfrac{1}{x^2+1}\right)$

(b) $f'(x) = \dfrac{(x^2-1)^4}{\sqrt{x^2+1}}\left(\dfrac{4}{x^2-1} - \dfrac{1}{x^2+1}\right)$

(c) $f'(x) = \dfrac{(x^2-1)^4}{\sqrt{x^2+1}}\left(\dfrac{4}{x^2-1} + \dfrac{1}{x^2+1}\right)$

(d) $f'(x) = \dfrac{(x^2-1)^4}{\sqrt{x^2+1}}\left(\dfrac{8}{x^2-1} - \dfrac{1}{x^2+1}\right)$

(e) $f'(x) = \dfrac{x(x^2-1)^4}{\sqrt{x^2+1}}\left(\dfrac{8}{x^2-1} - \dfrac{1}{x^2+1}\right)$

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4.(7 pts.) Compute the integral

$$\int_{2e}^{2e^2} \frac{1}{x\left(\ln\frac{x}{2}\right)^2}dx.$$

- Make the substitution $u = \ln\frac{x}{2}$ with $dx = xdu$. At $x = 2e$, have $u = 1$ and at $x = 2e^2$ have $u = 2$.
- $\displaystyle\int_{2e}^{2e^2} \frac{1}{x\left(\ln\frac{x}{2}\right)^2}dx = \int_1^2 \frac{1}{u^2}du = \left[-\frac{1}{u}\right]_1^2$
- $= -\frac{1}{2} + 1 = \frac{1}{2}.$

(a) 2 (b) $\frac{3}{2}$ (c) $\frac{1}{2}$ (d) 1 (e) 0

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5.(7 pts.) Which of the following expressions gives the partial fraction decomposition of the function

$$f(x) = \frac{x^2 - 2x + 6}{x^3(x-3)(x^2+4)}?$$

- We have x is a linear factor of multiplicity 3, $(x-3)$ is a linear factor of multiplicity 1 and (x^2+4) is an irreducible quadratic factor of multiplicity 1.
- $$\frac{x^2 - 2x + 6}{x^3(x-3)(x^2+4)} = \frac{A}{x^3} + \frac{B}{x^2} + \frac{C}{x} + \frac{D}{x-3} + \frac{Ex+F}{x^2+4}.$$

(a) $\dfrac{A}{x^3} + \dfrac{B}{x-3} + \dfrac{C}{x^2+4}$

(b) $\dfrac{A}{x^3} + \dfrac{B}{x-3} + \dfrac{Cx+D}{x^2+4}$

(c) $\dfrac{A}{x^3} + \dfrac{B}{x^2} + \dfrac{C}{x} + \dfrac{D}{x-3} + \dfrac{E}{x^2+4}$

(d) $\dfrac{A}{x^3} + \dfrac{B}{x^2} + \dfrac{C}{x} + \dfrac{D}{x-3} + \dfrac{Ex+F}{x^2+4}$

(e) $\dfrac{A}{x^3} + \dfrac{B}{x^2} + \dfrac{C}{x} + \dfrac{D}{x-3} + \dfrac{E}{x+2} + \dfrac{F}{x-2}$

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6.(7 pts.) Find $f'(x)$ if

$$f(x) = x^{\ln x} \ .$$

- One method is to use logarithmic differentiation. Let $y = f(x)$.
- $\ln y = \ln(x^{\ln x}) = (\ln x)(\ln x) = (\ln x)^2$.
- $\dfrac{y'}{y} = \dfrac{2\ln x}{x}$.
- Therefore $f'(x) = y' = 2(\ln x)x^{(\ln x)-1}$.
- **Alternatively** we have $f(x) = \left(e^{\ln x}\right)^{\ln x} = e^{(\ln x)^2}$
- $f'(x) = e^{(\ln x)^2}\dfrac{d(\ln x)^2}{dx} = e^{(\ln x)^2}(2\ln x)\dfrac{1}{x} = \dfrac{x^{\ln x}2\ln x}{x} = x^{(\ln x)-1}2\ln x$.

(a) $2(\ln x)x^{\ln x}$

(b) $x^{\ln x}\ln x$

(c) $2(\ln x)x^{(\ln x)-1}$

(d) $x^{\ln x}(\ln x + 1)$

(e) $x^{(\ln x)-1}\ln x$

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7.(7 pts.) Calculate the following integral.

$$\int_0^1 \frac{\arctan x}{1+x^2}\,dx\ .$$

- Make the substitution $u = \arctan x$ with $dx = (1+x^2)du$.
- $\displaystyle\int_0^1 \frac{\arctan x}{1+x^2}\,dx = \int_0^{\frac{\pi}{4}} u\,du$
- $\displaystyle = \left[\frac{u^2}{2}\right]_0^{\frac{\pi}{4}} = \frac{\pi^2}{32}.$

(a) $\frac{1}{2}$ (b) $\frac{\pi}{8}$ (c) $\frac{\pi^2}{32}$ (d) $\ln 2$ (e) $\frac{\pi^2}{8}$

8.(7 pts.) If 100 grams of radioactive material with a half–life of two days are present at day zero, how many grams are left at day three?

- We have initial amount $m_0 = 100$ and half life $t_{1/2} = 2$ days.
- The amount left after t days is given by $m(t) = m_0e^{kt} = 100e^{kt}$ for some constant k.
- To find the value of k, we use the fact that the half-life is 2 days. This tells us that $50 = 100e^{2k}$ or $\frac{1}{2} = e^{2k}$. Applying the natural logarithm to both sides, we get $\ln\frac{1}{2} = \ln e^{2k}$ or $-\ln 2 = 2k$.
- Therefore $k = \frac{-\ln 2}{2}$ and $m(t) = 100e^{-\frac{t\ln 2}{2}} = 100(e^{\ln 2})^{-\frac{t}{2}} = 100(2)^{-\frac{t}{2}}$
- After 3 days, we have $m(3) = 100(2)^{-\frac{3}{2}} = \frac{100}{3\sqrt{3}}$.

(a) $\frac{100}{4^{1/3}}$ (b) $\frac{100}{2^{1/3}}$ (c) $\frac{100}{\sqrt{2}}$ (d) $\frac{100}{\sqrt{8}}$ (e) 50

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9.(7 pts.) $\lim_{x\to 0^+} (\cos x)^{\frac{1}{x^2}} =$

- This is an indeterminate form of type 1^∞.
- We have
$$\lim_{x\to 0^+} (\cos x)^{1/x^2} = \lim_{x\to 0^+} Exp\left[\frac{\ln(\cos x)}{x^2}\right] = Exp\left[\lim_{x\to 0^+} \frac{\ln(\cos x)}{x^2}\right]$$
($Exp[x] = e^x$ here)
- $= \quad (byl'Hop) \quad Exp\left[\lim_{x\to 0^+} \frac{\frac{1}{\cos x}(-\sin x)}{2x}\right] = Exp\left[\lim_{x\to 0^+} \frac{-\tan x}{2x}\right]$
- $= \quad (byl'Hop) \quad Exp\left[\lim_{x\to 0^+} \frac{-\sec^2 x}{2}\right] = e^{-1/2}$

(a) e (b) $e^{-\frac{1}{2}}$ (c) ∞

(d) 1 (e) Does not exist

10.(7 pts.) The integral
$$\int_0^{\pi/2} x\cos(x)dx$$
is

- We use integration by parts with $u = x$, $dv = \cos x dx$. We get $du = dx$ and $v = \sin x$.
- Recall that $\int udv = uv - \int vdu$. Therefore $\int_0^{\pi/2} x\cos x dx = x\sin x\Big|_0^{\pi/2} - \int_0^{\pi/2} \sin x dx$
- $= \frac{\pi}{2}\sin\frac{\pi}{2} - 0 - \big[-\cos x\big]_0^{\pi/2} = \frac{\pi}{2} + \big[\cos\frac{\pi}{2} - \cos 0\big]$
- $= \frac{\pi}{2} + \big[0 - 1\big] = \frac{\pi}{2} - 1.$

(a) $\frac{\pi}{2} - 1$ (b) divergent (c) 0

(d) $\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}$ (e) $1 - \frac{\pi}{2}$

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11.(7 pts.) Evaluate the integral

$$\int_0^{\pi/2} \sin^3(x)\cos^5(x)dx.$$

- Use the identity $1-\cos^2(x) = \sin^2(x)$.
- Let $u = \cos(x),\ du = -\sin(x)dx$
- $\displaystyle\int_0^{\pi/2} \sin^3(x)\cos^5(x)dx = \int_0^{\pi/2} (1-\cos^2(x))\sin(x)\cos^5(x)dx = -\int_1^0 \left(u^5 - u^7\right)\,du.$
- $\displaystyle = \int_0^1 \left(u^5 - u^7\right)\,du = \left[\frac{u^6}{6} - \frac{u^8}{8}\right]_0^1$
- $\displaystyle = \frac{1}{6} - \frac{1}{8} = \frac{1}{24}.$

(a) $\frac{1}{4}$ (b) $\frac{1}{24}$ (c) 0 (d) $-\frac{1}{24}$ (e) $\frac{\pi}{2}$

12.(7 pts.) If you expand $\dfrac{2x+1}{x^3+x}$ as a partial fraction, which expression below would you get?

- $\displaystyle\frac{2x+1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$
- Multiplying the above equation by $x(x^2+1)$, we get $2x+1 = A(x^2+1) + x(Bx+C) = Ax^2 + A + Bx^2 + Cx = (A+B)x^2 + Cx + A$.
- Comparing coefficients, we get $A = 1,\quad C = 2$, and $A + B = 0$. Therefore $B = -A = -1$.
- The partial fractions decomposition of $\dfrac{2x+1}{x(x^2+1)}$ is therefore $\dfrac{1}{x} + \dfrac{-x+2}{x^2+1}$.

(a) $\dfrac{1}{x} + \dfrac{-x+2}{x^2+1}$ (b) $\dfrac{-2}{x} + \dfrac{1}{x^2+1}$

(c) $\dfrac{-1}{x} + \dfrac{x}{x^2+1}$ (d) $\dfrac{-1}{x^2} + \dfrac{1}{x+1}$

(e) $\dfrac{2}{x} + \dfrac{1}{x^2+1}$

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Partial Credit

You must show your work on the partial credit problems to receive credit!

13.(12 pts.) Find the integral

$$\int \frac{3x+1}{x^3+x^2}dx.$$

Solution: Use partial fraction decomposition

$$\frac{3x+1}{x^3+x^2} = \frac{3x+1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} = \frac{Ax(x+1)+B(x+1)+Cx^2}{x^2(x+1)}.$$

Therefore

$$3x+1 = (A+C)x^2 + (A+B)x + B.$$

It follows that

$$A+C=0, \quad A+B=3, \quad B=1,$$
$$A=2, \quad B=1, \quad C=-2,$$

and

$$\int \frac{3x+1}{x^3+x^2}dx = \int \Big(\frac{2}{x} + \frac{1}{x^2} - \frac{2}{x+1}\Big)dx = 2\ln|x| - \frac{1}{x} - 2\ln|x+1| + C.$$

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14.(12 pts.)
Evaluate

$$\int \frac{x^2}{\sqrt{9-x^2}}\, dx.$$

Note: The formula sheet will help you with this problem.

Write your answer in terms of the original variable x and (if needed) replace all composite trigonometric functions (such as $\cos(\sin^{-1}(x/n))$ etc...) by algebraic combinations of x.

- Here we use the trigonometric substitution $x = 3\sin\theta$, where $-\dfrac{\pi}{2} \leq \theta \leq \dfrac{\pi}{2}$.
- We have $x^2 = 9\sin^2\theta$, $\quad dx = 3\cos\theta d\theta$ and $\sqrt{9-x^2} = \sqrt{9 - 9\sin^2\theta} = 3|\cos\theta| = 3\cos\theta$
- $\int \dfrac{x^2}{\sqrt{9-x^2}}dx = \displaystyle\int \dfrac{9\sin^2\theta}{3\cos\theta} 3\cos\theta d\theta = 9\int \sin^2\theta d\theta.$
- $= \dfrac{9}{2}\displaystyle\int (1-\cos(2\theta))d\theta = \dfrac{9}{2}\left[\theta - \dfrac{\sin(2\theta)}{2}\right] + C$
- We have $\theta = \sin^{-1}\dfrac{x}{3}$. Therefore $\displaystyle\int \dfrac{x^2}{\sqrt{9-x^2}}dx = \dfrac{9}{2}\left[\sin^{-1}\left(\dfrac{x}{3}\right) - \dfrac{2\sin\theta\cos\theta}{2}\right] + C$
- Using a triangle, we get $\cos\theta = \dfrac{\sqrt{9-x^2}}{3}$ and

  $$\int \frac{x^2}{\sqrt{9-x^2}}dx = \frac{9}{2}\left[\sin^{-1}\left(\frac{x}{3}\right) - \frac{\frac{2}{9}\, x\sqrt{9-x^2}}{2}\right] + C$$

  $$= \frac{9}{2}\left[\sin^{-1}\left(\frac{x}{3}\right) - \frac{x\sqrt{9-x^2}}{9}\right] + C$$

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15. (6 pts.) Please circle "TRUE" if you think the statement is true, and circle "FALSE" if you think the statement is False.

(a)(1 pt. No Partial credit) $\lim_{x \to 0^+} \ln x = 0$.

TRUE <u>FALSE</u>

(b)(1 pt. No Partial credit) $\int \frac{1}{1+x^2} = \ln|1+x^2| + C$.

TRUE <u>FALSE</u>

(c))(1 pt. No Partial credit) $2^x = e^{x\ln(2)}$.

<u>TRUE</u> FALSE

(d))(1 pt. No Partial credit) In solving $\int \sqrt{x^2 - 4}\, dx$ with trigonometric substitution, the correct substitution to make is $x = 2\sin\theta$.

TRUE <u>FALSE</u>

(e))(1 pt. No Partial credit) If If $f(x) = \tan\left(\sin^{-1}\frac{x}{3}\right)$, then $f(x) = \frac{x}{\sqrt{9+x^2}}$ for any number x in the domain of f.

TRUE <u>FALSE</u>

(f))(1 pt. No Partial credit) $\ln\left(\frac{1}{x}\right) = -\ln(x)$ for all $x > 0$.

<u>TRUE</u> FALSE

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The following is the list of useful trigonometric formulas:

Note: $\sin^{-1} x$ and $\arcsin(x)$ are different names for the same function and $\tan^{-1} x$ and $\arctan(x)$ are different names for the same function.

$$\sin^2 x + \cos^2 x = 1$$

$$1 + \tan^2 x = \sec^2 x$$

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

$$\sin 2x = 2 \sin x \cos x$$

$$\sin x \cos y = \frac{1}{2}\big(\sin(x-y) + \sin(x+y)\big)$$

$$\sin x \sin y = \frac{1}{2}\big(\cos(x-y) - \cos(x+y)\big)$$

$$\cos x \cos y = \frac{1}{2}\big(\cos(x-y) + \cos(x+y)\big)$$

$$\int \sec\theta = \ln|\sec\theta + \tan\theta| + C$$

$$\int \csc\theta = \ln|\csc\theta - \cot\theta| + C$$

$$\csc\theta = \frac{1}{\sin\theta}, \quad \cot\theta = \frac{1}{\tan\theta}$$

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8.	(a)	(b)	(c)	(•)	(e)
9.	(a)	(•)	(c)	(d)	(e)
10.	(•)	(b)	(c)	(d)	(e)
11.	(a)	(•)	(c)	(d)	(e)
12.	(•)	(b)	(c)	(d)	(e)

Please do NOT write in this box.

Multiple Choice	______
13.	______
14.	______
15.	______
Total	______